13th FIG Symposium on Deformation Measurement and Analysis
4th IAG Symposium on Geodesy for Geotechnical and Structural Engineering

LNEC, LISBON 2008 May 12-15

BAYESIAN ESTIMATION IN DAM MONITORING NETWORKS

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Abstract: A Bayesian estimator with informative prior distributions (a multi-normal and an inverted gamma distribution), adequate to displacement estimation at dam displacement monitoring networks, is presented. The hyper-parameters of the prior distributions are obtained by Bayesian empirical methods with non-informative meta-priors. The performances of the Bayes estimator and the classical generalized lest squares estimator are compared using two measurements of the horizontal monitoring network of a concrete gravity dam: the Penha Garcia dam (Portugal). In order to test the robustness of the two estimators, a gross error is added to one of the measured horizontal directions: the Bayes estimator proves to be significantly more robust than the classic maximum likelihood estimator.

1. INTRODUCTION

Engineering networks established to monitor displacements of large dams, with behaviour analysis and safety control purpose, are usually tied to a local reference frame and are measured repeatedly over the dam's life span, which may last for many decades. The repeated observation of the networks provides the information necessary to "estimate" the hyper-parameters of the prior probability density functions (PDF) of the model's parameters, according to the so called parametric empirical Bayes (PEB) methods (Barnett, 1975; Carlin and Louis, 2000).

The stochastic models with different levels of distributions and parameters are called hierarchical Bayesian models (Press, 2003). The sampling distribution and the sampling PDF parameters occupy the first level. The prior and the posterior distributions and its hyper-parameters occupy the second level. The meta-prior and meta-posterior distributions and its meta-parameters occupy the third level. The empirical Bayesian inference on hyper-parameters, carried out at the second and third levels, precedes the Bayesian inference on sampling parameters, carried out at the first and second levels.

According to the Bayesian methodology, if the meta-priors are non-informative, such as improper uniform priors (Barnett 1975; Carlin and Louis, 2000), the maximization of the meta-posterior is equivalent to the maximization of the likelihood function. That is the option of this work: to use non-informative meta-priors. The choice of the priors (multi-normal and inverted gamma) is justified (Box and Tiao, 1992; Press, 2003) by mathematical convenience (the posteriors have multivariate t and inverted chi-square distributions) and agrees with the common sense. The estimation of the hyper-parameters is trivial, for the multi-normal prior, but not so for the inverted gamma prior (Casaca, 2007).

2. THE HIERARCHICAL MODEL

2.1. The Sampling Space

The first measurement of the network is to be carried out in a convenient epoch (t_0), when the dam reservoir is empty and the dam's internal temperature is extreme (low or high), so that the future estimated displacements may be easily compared with the dam's structural expected behaviour, allowing a rough quality control of the observations.

The functional model of a temporal engineering network that has been measured at the epochs (t_0, t_1,..., t_k) is the linear relation:

$$A\,X_i(n,1) = Y_i(m,1), \quad (i = 1, \ldots, k) \tag{1}$$

where: i) A is the network's first order design matrix; ii) X_i is the vector of the displacements of the network's vertices (object, ancillary and reference points), between the epoch t_0 and the epoch t_i; iii) Y_i is the vector of the changes of the observable variables (horizontal angles, distances, GPS baselines, etc.), between the epoch t_0 and the epoch t_i.

The matrix A, which results from the Taylorization of the non-linear relations between displacements of the network's vertices and the changes of the observable variables, according to the method of the variation of co-ordinates, may be computed with roughly approximate co-ordinates of the vertices, before the fieldwork, and may be kept constant over time.

The operational mathematical model results from setting up a stochastic model to the vector of the changes Y_i, in order to deal with the effect of the observation errors, which arise from several sources such as: faulty calibration of the measurement instruments, inadequate operative methods, adverse atmospheric conditions, operator's skill, etc.

A simple stochastic model consists of regarding the vector of changes Y_i as a sample of a multi-normal random vector, with mean vector $E(Y_i) = \mu_i$ and variance matrix $V(Y_i) = \omega_i\,\Sigma$, where ω_i is an unknown positive scalar parameter and $\Sigma(m, m)$ is the second order design matrix of the network: a known symmetric positive definite (spd) matrix.

The mean vector μ_i is the unknown vector of the changes of the observables that is related to the unknown vector of displacements θ_i by the linear relation $A\theta_i = \mu_i$.

The scalar parameter (ω_i) is a quality parameter that represents an improvement (if $\omega_i < 1$) or a degradation (if $\omega_i > 1$) of the network's second order design matrix (Σ), which is set up in the planning of the network and is used in the computation of error ellipses, etc. If the i^{th} measurement is carried out according to the plan, the quality parameter (ω_i) is expected to be one. In practice, due to changes in the measurement instruments, atmospheric conditions, operators, etc., the quality parameter (ω_i) varies randomly along the time.

2.2. The Parameter Space

The repeated measurement of the network along time (sometimes many decades) covers a wide range of actions on the dam (temperatures and water levels), with structural responses (displacements, etc.) that behave according to a visco-elastic model.

The distribution of the vectors of displacements (θ_1,..., θ_k) over time may be modeled by a multi-normal distribution $N(\mu_\theta, \Sigma_\theta)$, where the variance matrix (Σ_θ) represents the elastic response and the mean vector (μ_θ) represents a time effect. The multi-normal distribution $N(\mu_\theta, \Sigma_\theta)$

may be regarded as the mother population of the network's displacement vector parameter (θ). The correspondent PDF will be the prior PDF of the vector parameter (θ) and the mean vector μ_θ and the variance matrix Σ_θ will be two hyper-parameters of the hierarchical Bayesian model.

The parameters (ω_1,..., ω_k), that characterize the quality of the measurements, are independent of the actions on the dam and the correspondent displacements (θ). Their random distribution may be conveniently (Press, 2003) modeled with an inverted gamma distribution, which depends of a shape parameter (α) and a scale parameter (β) (Casaca, 2007).

The parameter space of the Bayes hierarchical model is defined by two prior PDF and four hyper-parameters: i) The multi-normal prior PDF of the displacements vectors:

$$h_\theta(\theta\,|\,\mu_\theta,\Sigma_\theta) = \frac{exp\left(-\frac{1}{2}(\theta-\mu_\theta)^T\,\Sigma_\theta^{-1}(\theta-\mu_\theta)\right)}{\sqrt{(2\pi)^m\,det(\Sigma_\theta)}} \tag{2}$$

with the hyper-parameters (μ_θ, Σ_θ); ii) The inverted gamma prior PDF of the quality parameters:

$$h_\omega(\omega\,|\,\alpha,\beta) = \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)\omega^{-(\alpha+1)}\,exp\left(-\frac{\beta}{\omega}\right) \tag{3}$$

with the hyper-parameters (α, β).

3. BAYES ESTIMATORS VERSUS CLASSICAL ESTIMATORS

For a given vector of observables Y*, the Bayes estimates of the parameters (θ, ω) must minimize the posterior PDF of the parameters:

$$g(\theta,\omega\,|\,Y^*) \propto L(\theta,\omega\,|\,Y^*)\,h_\theta(\theta)\,h_\omega(\omega) \tag{4}$$

which is proportional (∝) to the product of the likelihood L(θ, ω|Y*) and the joint prior PDF of the parameters $h(\theta, \omega) = h_\theta(\theta)\, h_\omega(\omega)$, since θ and ω are supposed to be stochastically independent.

3.1. Non Informative Priors

If the priors h_θ and h_ω are non-informative priors, such as uniform distributions, the Bayes solutions coincide with the classical maximum likelihood (ML) solutions. In this case, the vector parameter that maximizes (4) is the best linear unbiased estimator (BLUE):

$$\theta_{ML} = (A^T\Sigma^{-1}A)^{-1}A^T\Sigma^{-1}Y^* \tag{5}$$

The scale parameter that maximizes the posterior PDF is also the ML solution:

$$\omega_{ML} = \frac{1}{m}(Y^*-A\theta_{ML})^T\,\Sigma^{-1}(Y^*-A\theta_{ML}) \tag{6}$$

which must be computed with the BLUE (5). In practice, as the ML solution (6) is biased, a more popular, and unbiased, solution is:

$$\omega_0 = \frac{1}{m-n}(Y^* - A\theta_{ML})^T \Sigma^{-1}(Y^* - A\theta_{ML}) \tag{7}$$

3.2. Informative Priors

If the priors h_θ and h_ω are informative, the Bayes solutions do not coincide any more with the classical maximum likelihood solutions. If the prior PDF h_θ belongs to a multi-normal family and the prior PDF h_ω belongs to an inverted gamma family, the Bayes solutions may become quite different from the classical ML solutions.

In this case (Box and Tiao, 1992; Casaca, 2007), the vector parameter that maximizes the posterior PDF (4) is:

$$\theta_B = (\omega_B^{-1} A^T \Sigma^{-1} A + \Sigma_\theta^{-1})^{-1}(\omega_B^{-1} A^T \Sigma^{-1} Y^* + \Sigma_\theta^{-1}\mu_\theta) \tag{8}$$

and the quality parameter that maximizes the posterior PDF (4) is:

$$\omega_B = \frac{2\beta}{m+2(\alpha+1)}(Y^* - A\theta_B)^T \Sigma^{-1}(Y^* - A\theta_B) \tag{9}$$

which should be replaced by the unbiased alternative solution:

$$\omega_B = \frac{2\beta}{m-n+2(\alpha+1)}(Y^* - A\theta_B)^T \Sigma^{-1}(Y^* - A\theta_B) \tag{10}$$

The Bayes solutions (θ_B and ω_B) are mutually dependent and must be determined simultaneously. One possible computational strategy is based on an iterative procedure of the kind: i) Start with an unit scale parameter ($\omega_{B0} = 1$), compute the vector parameter (θ_{B0}) with (8) and replace it at (10) to compute a new scale parameter (ω_{B1}); ii) Repeat the procedure until both estimates (θ_{Bk} and ω_{Bk}) become stable.

4. THE PARAMETRIC EMPIRICAL APPROACH

After the choice of the priors h_θ and h_ω, it is necessary to choose their hyper-parameters. Whenever there is data from previous experiments similar to the current one, the parametric empirical approach is recommendable (Barnett, 1973). Two situations may arise: i) There is prior information on the distribution of the hyper-parameters (meta-priors and meta-parameters); ii) There is no prior information on the distribution of the hyper-parameters (the meta-priors are uniform). In the second hypothesis the Bayesian approach coincides with the classical maximum likelihood approach.

4.1. The Displacement Vectors

Let (X_1,..., X_k) to be stochastically independent estimates of the vector parameters (θ_1,..., θ_k), which are supposed to belong to a multi-normal family $N(\mu_\theta, \Sigma_\theta)$. If there is no prior information on the distribution of the hyper-parameters (μ_θ, Σ_θ), the maximum likelihood solutions of the problem (Morrison, 1990) are the empirical mean vector:

$$m_\theta = \frac{1}{k}\sum_{i=1}^{k} X_i \tag{11}$$

and the empirical variance matrix:

$$S_\theta = \frac{1}{k}\sum_{i=1}^{k}(X_i - m_\theta)(X_i - m_\theta)^T \tag{12}$$

The maximum likelihood estimator (12) of the variance matrix (Σ_θ) is biased. However, a minor modification of (12) provides an unbiased estimator (Morrison, 1990):

$$S_\theta = \frac{1}{k-1}\sum_{i=1}^{k}(X_i - m_\theta)(X_i - m_\theta)^T \tag{13}$$

4.2. The Quality Parameters

Let (s_1,..., s_k) to be stochastically independent estimates of the quality parameters (ω_1,..., ω_k), which are supposed to belong to an inverted gamma family with hyper-parameters (α, β). If there is no prior information on the distribution of the hyper-parameters, the maximum likelihood solutions of the problem (Casaca, 2007) are given by the resolution, in order to the hyper-parameter (α), of the equation:

$$\psi(\alpha) - ln(\alpha) = ln\left(\sum_{i=1}^{k}\frac{1}{s_i}\right) - ln(k) - \frac{1}{k}\sum_{i=1}^{k} ln(s_i) \tag{14}$$

where $\psi(\alpha)$ is the digamma function (Casaca, 2007). The shape hyper-parameter (α) may be then inserted in:

$$\beta = k\,\alpha\left(\sum_{i=1}^{k}\frac{1}{s_i}\right)^{-1} \tag{15}$$

to solve for the scale hyper-parameter (β).

5. THE PENHA GARCIA DAM NETWORK

The Penha Garcia dam is a concrete gravity dam, with a maximum height of 25m and a crest length of 112m, on the river Ponsul, 60km to the Northeast of the city of Castelo Branco in Portugal. Its geodetic surveying system consists of a precision geometric leveling line installed on the dam's crest and a rudimentary triangulation network, with two station points, on the downstream banks, and several object points on the dam (Figure 1). According to a prior geotechnical evaluation, the station points (PD and PE) are supposed to be stable with time and, therefore, materialize the network's local datum.

From each station point, taking the other station point as origin, two independent arcs of horizontal directions are measured, with precision electronic theodolites. Between 1981 and 2007, twenty three measurements of the network were carried out. The 23rd measurement was preserved, to be processed with the Bayes estimators built with priors derived from the previous 22 measurements.

Taking the initial measurement as a reference, twenty one vectors of changes (Y_i) were computed with the BLUE:

$$X_i = (A^T \Sigma^{-1} A)^{-1} A^T \Sigma^{-1} Y_i \quad (i = 1, \ldots, 21) \tag{16}$$

The 21 estimates of the displacement vectors ($X_1, \ldots, X_{21}$) were used to compute, with the relations (11) and (12), the hyper-parameters (m_θ, S_θ) of the prior multi-normal distribution of the displacement vectors. The components of the resulting mean vector (m_θ) are plotted on the object points (6D, 5D, 4D, 4E, 3E, 2E) at the Figure 1. The dispersion ellipses, for the 0.95 probability level, derived from the variance matrix S_θ of the prior PDF h_θ, were computed and plotted with the center on the object points (larger ellipses of the Figure 1). The error ellipses, for the 0.95 probability level, derived from the variance matrix Σ (second order design matrix of the network), were also computed and plotted with the center on the object points (minor ellipses of the Figure 1).

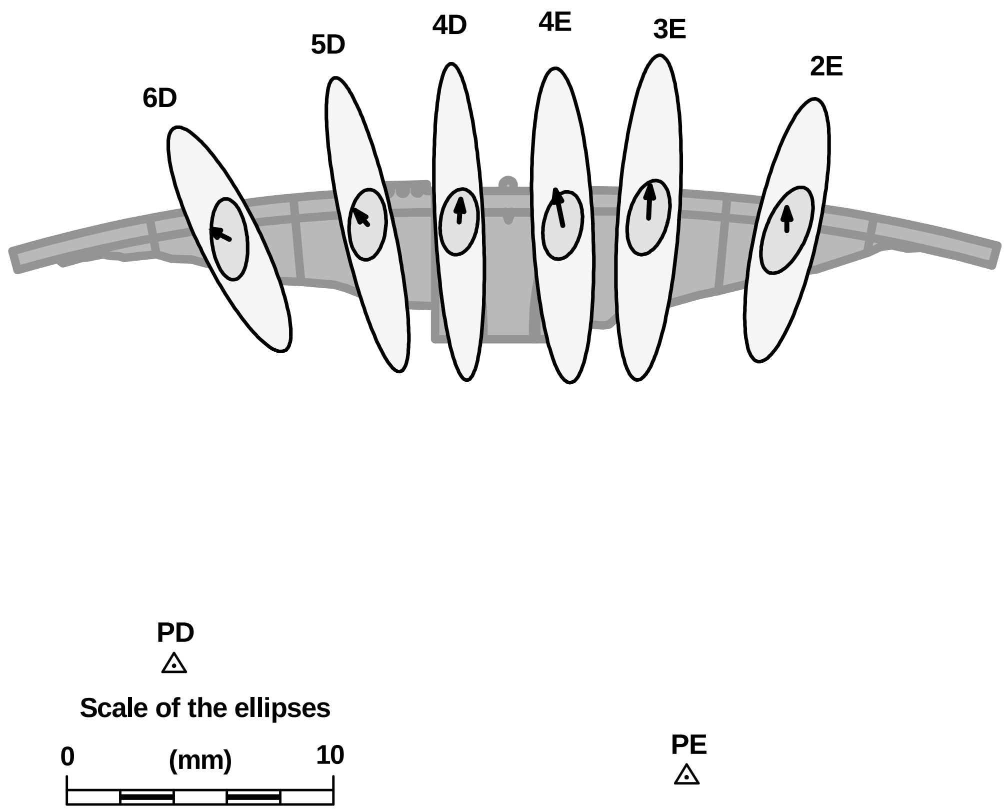

Figure 1 – Penha Garcia dam network: station points (PD and PE) and object points with dispersion and error ellipses.

The analysis of the Figure 1 shows that: i) The components of the hyper-parameter m_θ lie within the network's error ellipses, meaning that, after a quarter of century of the dam's operation, the time effect is not very significant; ii) The error ellipses lie within the dispersion ellipses in a proper way, meaning that the network is adequate to measure the dam's expected displacements.

The 21 estimates of the quality parameter ($s_1,\ldots, s_{21}$) presented at the Table 1, were used to compute, with the relations (14) and (15), the hyper-parameters (α = 3.60, β = 2.46) of the prior inverted gamma distribution of the quality parameters.

Table 1 – Estimates (s) of the quality parameter (ω)

year	s	year	s	year	s	year	s	year	s	year	s	year	s
1982	0.37	1985	1.43	1988	0.59	1991	1.00	1994	3.48	1998	0.91	2004	0.68
1983	0.24	1986	1.20	1989	0.97	1992	1.96	1995	0.66	2002	0.63	2005	1.53
1984	0.95	1987	0.60	1990	1.81	1993	0.35	1996	0.95	2003	0.71	2006	0.36

6. CLASSICAL VERSUS BAYESIAN INFERENCE

The dam's network 23rd measurement – carried out in June of 2007 – was processed with the classical maximum likelihood (ML) estimators (5) and (7) and with the Bayes estimators (8) and (10), with the priors h_θ (2) and h_ω (3) with the empirical hyper-parameters (m_θ, S_θ, α, β), derived from the previous 22 measurements. The Table 2 presents the ML solutions (dx, dy) and the Bayes solutions (dx, dy), which are very likely. The Figure 2 presents these two solutions (displacements) with the same symbol: a white circle.

Table 2 – Maximum likelihood (ML) and Bayes solutions.

	ML				Bayes			
Point	**dx**	**dy**	**dx***	**dy***	**dx**	**dy**	**dx***	**dy***
	(mm)	(mm)	(mm)	(mm)	(mm)	(mm)	(mm)	(mm)
2E	– 0.4	+ 3.4	– 0.4	+ 3.4	– 0.2	+ 3.5	– 0.1	+ 3.0
3E	– 0.5	+ 4.4	– 0.5	+ 4.4	– 0.4	+ 4.4	– 0.1	+ 3.8
4E	– 0.9	+ 4.3	– 2.8	+ 1.8	– 0.8	+ 4.3	– 1.1	+ 3.6
4D	– 0.3	+ 3.5	– 0.3	+ 3.5	– 0.2	+ 3.5	– 0.1	+ 3.2
5D	– 1.5	+ 2.9	– 1.5	+ 2.9	– 1.4	+ 3.0	– 1.2	+ 2.5
6D	– 2.2	+ 2.0	– 2.2	+ 2.0	– 2.2	+ 2.1	– 1.7	+ 1.3

To compare the performances of the classical (ML) and the Bayes estimators in face of contaminated observations, a gross error of 5 mgon was introduced in one of the two directions measured from the station PE to the object point 4E (Figure 1). The 23rd measurement contaminated with the gross error was processed with the classical maximum likelihood (ML) estimators (5) and (7) and with the Bayes estimators (8) and (10) with the same priors h_θ (2) and h_ω (3), with the same empirical hyper-parameters (m_θ, S_θ, α, β). The Table 2 presents the ML solutions (dx*, dy*) and the Bayes solutions (dx*, dy*) that resulted from the adjustment of the contaminated observables vector.

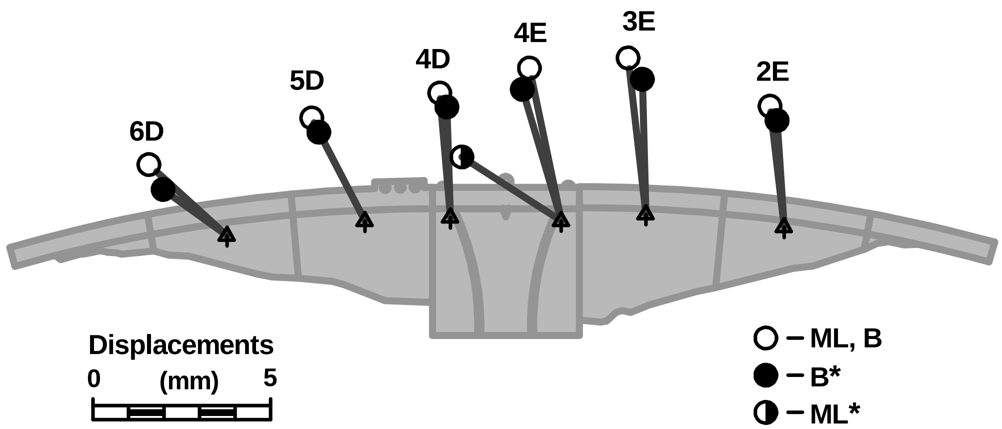

Figure 2 – Penha Garcia dam network: maximum likelihood (ML) and Bayes (B) solutions. The asterisk indicates the ML and B solutions in the contaminated case.

The analysis of the Table 2 shows that the two ML solutions (for the uncontaminated and the contaminated observables), with the exception of the point 4E, are similar. They are represented in the Figure 2 by the white circle symbol, except the 4E ML solution that is represented with a black and white circle symbol. The Bayes solutions for the contaminated observables – represented with the black circle symbol at the Figure 2 – are similar to the ML and Bayes solutions for the uncontaminated observables. The Bayes solution of the object point 4E is not affected by the contamination.

The analysis of the residuals of the two estimates is enlightening: i) In the case of the maximum likelihood estimate, the 5mgon contamination error originates two ambiguous symmetrical residuals of +2.8mgon and – 2.8mgon, respectively, in the two measured directions from the station PE to the point 4E; ii) In the case of the Bayes estimate, the 5mgon contamination error originates an unequivocal large residual of +4.8mgon, in the contaminated measurement.

One last note: the hyper-parameter (m_θ) estimated for the multi-normal prior, is not significant (according to the Figure 1, its components lie inside the network's error ellipses). If (μ_θ) is replaced by the null vector in the relation (8) the resulting Bayes estimates are scarcely different.

7. CONCLUSIONS

The Bayes estimator with PEB priors has a performance which is rather attractive, when compared to the usual maximum likelihood estimator: in normal circumstances, the results are very similar, but in face of contamination with gross errors, the Bayes estimator proves to be more robust than the maximum likelihood estimator.

In the case of networks with PEB priors consolidated by experience, it appears to be a good strategy to use simultaneously the maximum likelihood and the Bayes estimators. The similarity of the two solutions indicates a normal situation, but any discrepancy between the two solutions indicates that the observations do not agree with the previous experience and recommends further analysis of the available data.

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