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COMPARISON OF MONTE CARLO AND FUZZY TECHNIQUES IN UNCERTAINTY MODELLING

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Abstract: The standard reference in uncertainty modelling is the "Guide to the Expression of Uncertainty in Measurement (GUM)". GUM groups the occurring uncertain quantities into "Type A" and "Type B". Uncertainties of "Type A" are determined with the classical statistical methods, while "Type B" is subject to other uncertainties like experience with and knowledge about an instrument. Both types of uncertainty can have random and systematic error components. Our study focuses on a critical comparison of Monte Carlo (MC) and Fuzzy techniques in the propagation process of the different uncertainties, especially those of "Type B". Whereas MC techniques treat all uncertainties as having a random nature, the Fuzzy technique distinguishes between random and systematic errors. The random components are modelled in a stochastic framework, and the systematic uncertainties were treated with Fuzzy techniques. The applied procedure is outlined showing both the theory and a numerical example for the evaluation of uncertainties in an application for laserscanning.

1. INTRODUCTION

The "Guide to the Expression of Uncertainty in Measurement (GUM)" is the standard reference in uncertainty modelling in engineering and mathematical science, cf. (ISO, 1995). GUM groups the occurring uncertain quantities into "Type A" and "Type B". Uncertainties of "Type A" are determined with the classical statistical methods, while "Type B" is subject to other uncertainties like experience with and knowledge about an instrument. Whereas the uncertainties of the uncertain quantities of "Type A" can be estimated based on the measurement itself, the estimated uncertainties of the uncertain quantities of "Type B" are based on expert knowledge, e.g., the technical knowledge about an instrumental error source. Both types of uncertainty can have random and systematic error components:

- A random error ε arises from non predictable variations of some influence factors under seemingly the same actual conditions (non reproducible effects), see, e.g., Bandemer (2006, pp. 63ff).
- A systematic error δ is due to non controllable effects during the measurement and the preprocessing steps of the measurement, it biases the output quantity y. Although systematic errors are unknown, they bias the measurement result in one direction (reproducible, but unknown effects).

GUM defines an output quantity y as a function of input quantities $\boldsymbol{z}$ (preprocessing steps):

$$y = f(z_1, z_2, ..., z_n) = f(\boldsymbol{z}), \quad (1)$$

with n the number of input quantities $\boldsymbol{z}$, which can be a quantity (ISO 1995, chapter 4.1.3):

- "…, whose values and uncertainties are directly determined in the current measurement (original measurement)."
- "…, whose values and uncertainties are brought into the measurement from external sources, like the values from a calibration for an instrument (influence factor)."

Please note that in general the input quantities z_i may be a measurement result y itself. In order to have a clear representation, only the case where z_i is a measurement or an influence factor is treated in this paper. The quantity z_i can be carrier of both, random and systematic errors. GUM proposes to treat both errors (random and systematic) in a stochastic framework and introduces variances to describe their uncertainties.

Let us denote the function $f(...)$ from Eq. (1) as observation model and divide the influence quantities into three groups: additional information, sensor parameters, and model constants. Whereas the uncertainty of the original measurement is usually of "Type A", the uncertainty of the influence factors can be of "Type A" or "Type B". Fig. 1 shows the interaction between the measurement, the influence factors and the observation model. Systematic errors of the input quantities are meaningful by many reasons:

- The model constants are only partially representative for the given situation (e. g., the model constants for the refraction index for distance measurements).
- The number of additional information (measurements) may be too small to estimate reliable distributions for a random treatment.
- Measurement results are affected by rounding errors
- Other non-random errors of the output quantity occur due to neglected correction and reduction steps and for effects that cannot be modelled.

The paper is organized as follows: First we will describe the general idea of Monte Carlo techniques to describe measurement uncertainties in the context of GUM; second a Fuzzy approach to handle these measurement uncertainties is introduced. Then both approaches are applied to laserscanning and the obtained results are critically compared to each other. The paper finishes with a discussion and an outlook for further research.

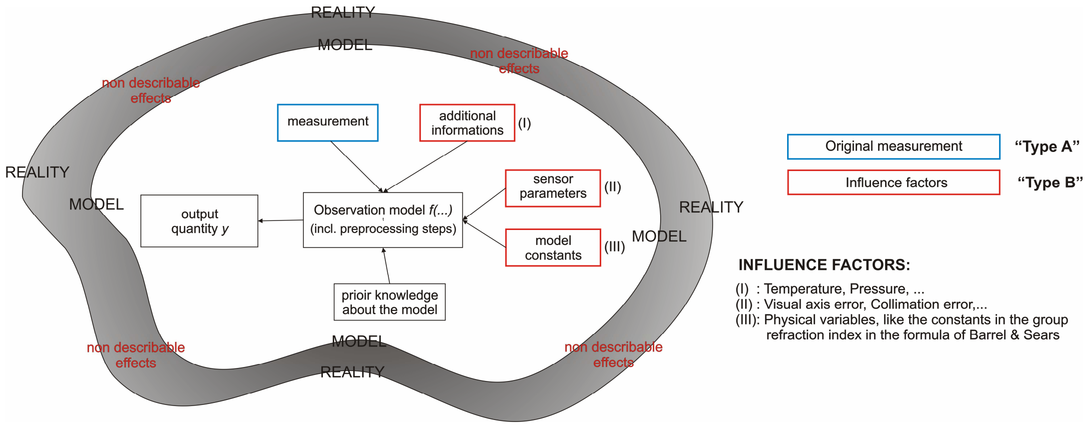

Figure 1 - Interaction between input quantities, the observation model and the output quantities

2. UNCERTAINTY MODELLING WITH MONTE CARLO TECHNIQUES

In Monte Carlo (MC) techniques, both, the random and the systematic components of the uncertainty are treated as having a random nature. Please note that not the systematic component itself is modelled as random, it is the knowledge about the systematic component for which a probability distribution is introduced (Koch, 2007).

The GUM suggested in some cases to select the probability distribution function (pdf) of the input quantities as rectangular, triangular, and trapezoidal (ISO, 1995). In these cases, it is hard/impossible to obtain the estimate of the uncertainty for the output quantity in a closed mathematical form. An alternative to modelling and propagating uncertainties is propagating distributions by MC simulations of the observation model from Eq. (1):

$$Y = f(Z_1, Z_2, ..., Z_n) = f(\mathbf{Z}). \tag{2}$$

Here Y represents a random output quantity and $Z_1, Z_2, ..., Z_n$ are the n random inputs.

2.1. Monte Carlo Approach to Evaluate Uncertainty

The MC techniques are of great importance for uncertainty evaluation. With a set of generated samples the distribution function for the value of the output quantity Y in (2) will be numerically approximated. MC approaches to estimate the uncertainty include the following steps:

- A set of random samples, which have the size n, is generated from the (pdf) for each random input quantity $Z_1, Z_2, ..., Z_n$. The sampling procedure is repeated M times for every input quantity.
- The output quantities Y will be then calculated by:

  $$y^{(i)} = f(z_1^{(i)}, z_2^{(i)}, ..., z_n^{(i)}) = f(\mathbf{z}^{(i)}), \tag{3}$$

  with the $i = 1...M$ generated samples of Y, we obtain an estimate of the pdf for Y.
- Particularly relevant estimates of any statistical quantities can be calculated:

  1) The expectation of the output quantity:

$$\hat{E}(y) = \frac{1}{M}\sum_{i=1}^{M} f(\mathbf{z}^{(i)}) \quad (4)$$

2) The estimate of the variance of the output quantity (Alkhatib, 2007):

$$\hat{\sigma}_y^2 = \frac{1}{M}\sum_{i=1}^{M}(f(\mathbf{z}^{(i)}) - \hat{E}(f(\mathbf{z})))(f(\mathbf{z}^{(i)}) - \hat{E}(f(\mathbf{z})))^T \quad (5)$$

3) The confidence interval $y_{conf,MC} = [\underline{y}, \overline{y}]$ of the estimate of the output quantity with the significance level of γ. To compute the confidence interval by MC simulation, one has to order the independent samples y from the smallest to largest, an approximate $100 \cdot (1\text{-}2\gamma)\%$ for the random variable Y is given by (Buckland, 1983):

$$y_{conf,MC} = [\underline{y} = y_j, \overline{y} = y_k], \text{ where } j = (M+1)\gamma \text{ and } k = (M+1)(1-\gamma). \quad (6)$$

2.2. Sampling from Probability Distribution Function

Any MC simulation requires random numbers. Random numbers are generated on a computer by means of deterministic procedures. In particular, rectangular distributed random numbers are generated, which may then in turn be transformed into random numbers of random variables having other distributions, for instance, into numbers of a normally distributed random variable (Gentel, 2003).

To demonstrate the modelling of uncertainty with a MC simulation in section 4, the generation algorithms of random numbers from rectangular, triangular and normal distribution will be shortly described. For more details, see, e.g., Koch (2007):

- *Generation of rectangular-distributed random numbers*:

  1) Generate $x_1, x_2, ..., x_n$ realisations of random variables that have the rectangular distribution on the unit interval [0, 1].
  2) Then $y = a_- + (a_+ - a_-) \cdot x$ is rectangular-distributed on the interval $[a_-, a_+]$. Here a_-, a_+ are the distribution parameter.

- *Generation of triangular-distributed random numbers:* The symmetric triangular distribution with pdf is of the form:

$$p(x \mid a_-, a_+) = \begin{cases} \dfrac{x - a_-}{a^2} & \text{for } a_- \le x \le a_- + a \\ \dfrac{a_+ - x}{a^2} & \text{for } a_- + a \le x \le a_+ \end{cases} \quad \text{with} \quad a = (a_+ + a_-)/2 \quad (7)$$

  The inverse cumulative density function (cdf) approach is used to generate random numbers $y_1, y_2, ..., y_n$ from the triangular distribution, cf., e.g., (Gentel, 2003, pp. 102):

  1.) Generate the random value y for the random variable from rectangle distribution $Y : U(0,1)$.
  2.) Set y equal to the distribution function, that is: $F(x) = h$.

3.) Invert the distribution function and isolate x, that is: $x = F^{-1}(h)$

4.) Calculate $x_i = F^{-1}(h_i)$ from inverse CDF:

$$F^{-1}(h) = \begin{cases} \sqrt{h(a_+ - a_-)(a - a_-)} + a_- & \text{if} \quad \sqrt{h(a_+ - a_-)(a - a_-)} + a_- < a \\ a_+ - \sqrt{(1-h)(a_+ - a_-)(a_+ - a)} & \text{else} \end{cases} \tag{8}$$

- *Generation of correlated normally-distributed random numbers:* It is well known that the multinormal distribution is fully characterized by its expected value $\boldsymbol{\mu}$ and its variance-covariance matrix $\boldsymbol{\Sigma}$. To generate random numbers from the multinormal distribution, the following steps have to be performed, see, e.g., (Gentle 2003, p. 197):

  1.) Compute the Cholesky decomposition, that is: $\boldsymbol{\Sigma} = \mathbf{R}^{\mathbf{T}}\mathbf{R}$.

  2.) Generate a realisation of an independent and normal random vector $\boldsymbol{Z} \sim N(\mathbf{0}, \mathbf{I})$.

  3.) Compute the transformed $\boldsymbol{Y} = \mathbf{R}^{\mathbf{T}}\boldsymbol{Z}$.

  4.) Compute transformed realisations according to $\boldsymbol{Y} = \boldsymbol{\mu} + \boldsymbol{Z}$.

  5.) The vector $\boldsymbol{Y}$ is $\boldsymbol{Y} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distributed.

3. A FUZZY APPROACH TO UNCERTAINTY MODELLING

In this section, a Fuzzy approach to uncertainty modelling in the context of GUM is introduced. Fuzzy theory (Zadeh 1965) has proven to be an appropriate solution for the description of uncertainties. Recently, many procedures have been introduced in different engineering applications, cf., e. g., (Ferson et al., 2002; Möller and Beer, 2004), incl. discussions about combined approaches in Fuzzy theory, interval mathematics and probability theory (Ferson et al., 2002).

In the here presented approach we distinguish between random and systematic errors in the propagation process of the uncertainties of the input quantities $\boldsymbol{z}$ to the output quantity y. Whereas the random part is treated with the law of propagation of covariances or with the MC approach, systematic errors are propagated within a sensitivity analysis (see section 3.2). Both types of uncertainty are modelled in a comprehensive way, using fuzzy intervals (see section 3.1). This procedure is in full accordance with the recommendations in the GUM; the difference is in the treatment of the systematic errors, for which no variances are introduced.

3.1. Uncertainty Modelling using Fuzzy Intervals

The random and systematic components of the uncertainties are characterized with a special case of Fuzzy theory, so called *Fuzzy Randomness* (Möller and Beer 2004; Viertl 1996). Each uncertain quantity z_i is exclusively modelled in terms of fuzzy intervals. A fuzzy interval $\tilde{A}$ is uniquely defined by its membership function $m_{\tilde{A}}(x)$ over the set $\mathbb{R}$ of real numbers with a membership degree between 0 and 1:

$$\tilde{A} := \{(x, m_{\tilde{A}}(x)) \,|\, x \in \mathbb{R}\} \qquad \text{with} \qquad m_{\tilde{A}} : \mathbb{R} \to [0,1] \tag{9}$$

The membership function of a fuzzy interval can be described by its left (*L*) and right (*R*) reference function (see also Fig. 2):

$$m_{\tilde{A}}(x) = \begin{cases} L\left(\dfrac{x_m - x - r}{c_l}\right), & \text{for} \quad x < x_m - r \\ 1, & \text{for} \quad x_m - r \le x \le x_m + r \\ R\left(\dfrac{x - x_m - r}{c_r}\right), & \text{for} \quad x > x_m + r \end{cases} \quad (10)$$

with x_m denoting the midpoint, r its radius, and c_l, c_r the spread parameters of the monotonously decreasing reference functions (convex fuzzy intervals).

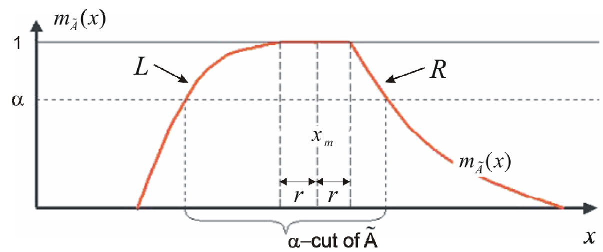

Figure 2 - Fuzzy interval and its α-cut

Fuzzy intervals serve now as basic quantities; their midpoints x_m are considered in the following as random variables and their spread describes the range of the systematic errors. The construction of the membership function is based on expert knowledge or knowledge about an instrument. In contrast to the MC approach, the membership function of a fuzzy interval cannot be interpreted in a probabilistic meaning and therefore the propagation of the systematic uncertainties has to be modified (see section 3.2). In the fuzzy case, we model the systematic component of the uncertainty itself and not the knowledge about the systematic component like it is in the MC approach. The $\alpha - cut$ with $\alpha \in [0,1]$ of a fuzzy interval $\tilde{A}$ is defined by:

$$\tilde{A}_\alpha := \left\{ x \in X \,\middle|\, m_{\tilde{A}}(x) \ge \alpha \right\}. \quad (11)$$

Each $\alpha - cut$ represents in case of monotonously decreasing reference functions a classical interval. The lower $\tilde{A}_{\alpha,\min}$, and the upper bound $\tilde{A}_{\alpha,\max}$ of an $\alpha - cut$ and its radius $\tilde{A}_{\alpha,r}$ are:

$$\tilde{A}_{\alpha,\min} = min\left(\tilde{A}_\alpha\right), \ \tilde{A}_{\alpha,\max} = max\left(\tilde{A}_\alpha\right) \text{ and } \tilde{A}_{\alpha,r} = \left(\tilde{A}_{\alpha,\max} - \tilde{A}_{\alpha,\min}\right)/2 \quad (12a,b,c)$$

The integral over all α-*cuts* equals the membership function:

$$m_{\tilde{A}}(x) = \int_0^1 m_{\tilde{A}_\alpha}(x)\, d\alpha \quad (13)$$

3.2. Uncertainty Propagation within a Sensitivity Analysis

The propagation process of the random and systematic errors is separated in two parts. Whereas the random components are treated with the law of variance propagation (GUM, chapter 5.2) or within a MC approach (see section 2.1), the propagation of the systematic errors is a range-of-values search problem. The propagation process leads to a fuzzy interval for the output quantity $y \rightarrow m_{\tilde{y}}(x)$. The approximate midpoint of the fuzzy interval for the output quantity y_m is:

$$y_m = f(z_{1_m}, z_{2_m}, \ldots, z_{n_m}) = f(\mathbf{z}_m)\,. \tag{14}$$

The computation of the membership function for the measurement results is based on the *α-cuts* $\tilde{z}_\alpha$ of the input quantities, within an optimization problem of the following target function, see, e.g., Kutterer and Neumann (2007):

$$\tilde{y}_{\alpha,\min} = \min_{z_i \in [\tilde{z}_{\alpha,\min}, \tilde{z}_{\alpha,\max}]} f(\boldsymbol{z}) \qquad \text{and} \qquad \tilde{y}_{\alpha,\max} = \max_{z_i \in [\tilde{z}_{\alpha,\min}, \tilde{z}_{\alpha,\max}]} f(\boldsymbol{z})\,. \tag{15}$$

The membership function of the output quantity is constructed based on a sufficient number of *α-cuts* from Eq. (15):

$$m_{\tilde{y}}(x) = \int_0^1 m_{\tilde{y}_\alpha}(x)\,d\alpha \qquad \text{with} \qquad m_{\tilde{y}_\alpha} = \left[\tilde{y}_{\alpha,\min}, \tilde{y}_{\alpha,\max}\right]. \tag{16}$$

In case of linear reference functions for the membership function of the input quantities, the propagation of systematic errors needs only be applied for the *α-cuts* with $\alpha = 0$ and $\alpha = 1$.

Finally, the confidence interval $y_{conf,Fuzzy}$ in the fuzzy case (at the α-level) is then obtained by the combination of both uncertainty components:

$$y_{conf,Fuzzy} = [\underline{y} - \tilde{y}_{\alpha,r}\ ;\ \overline{y} + \tilde{y}_{\alpha,r}]\,. \tag{17}$$

Whereas the α-level of zero corresponds to the pessimistic case, the optimism case is obtained for $\alpha = 1$. Only the random uncertainty component from the input quantities $\boldsymbol{z}$ contributes to the lower and upper bound of the MC confidence interval $y_{conf,MC} = [\underline{y}, \overline{y}]$.

4. NUMERICAL EXAMPLE FOR AN APPLICATION TO LASERSCANNING

In this section a short numerical example for the comparison of the two approaches from section 2 and 3 is presented. The aim is to detect the vertical displacements of the bridge under load, e. g., due to traffic or train crossings (Strübing 2007). For this reason, a laserscanner of type *Leica HDS 4500* was placed beneath the bridge; the measurements in the "Profiler Mode" span the green plane in Fig. 3. The discrepancies to the standard case of normal distributed measurements are meaningful by many reasons (see also section 1): The laserscanner carries out very fast measurements and the measurements are influenced by vibrations due to the traffic load of the bridge.

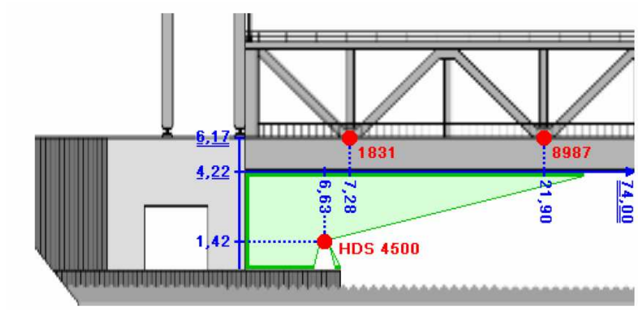

Figure 3 - Position of the laserscanner beneath the bridge (Strübing, 2007)

The time series of the vertical height $h_{scan}(t)$ of the bridge at the stations 7.28 m and 21.90 m can be expressed in the local coordinate system of the laserscanner by the following equation:

$$h_{scan}(t) \; = \; s_{sl}(t) \cdot \cos\left(z(t)\right), \tag{18}$$

with the slope distance $s_{sl}(t)$ and the zenith angle $z(t)$, measured by the laserscanner. The number of measured epochs q is 100. The vertical displacements $w_{scan}(x,t)$ of the bridge are obtained by subtracting the mean height of the bridge from the time series in Eq. (18):

$$w_{scan}(t) = \; h_{scan}(t) - \frac{1}{q}\sum_{t=0}^{q} h_{scan}(t) = s_{sl}(t) \cdot \cos\left(z(t)\right) - \frac{1}{q}\sum_{t=0}^{q} s_{sl}(t) \cdot \cos\left(z(t)\right) \tag{19}$$

4.1. Uncertainties for the measurements and influence factors

The output quantity $y \,@ w_{scan}(x,t)$ depends on the following input quantities z_i:

- Accuracy of the distance (z_1, Type A), and their additional constant (z_2, Type B)
- Distance depending term for the accuracy of the distance measurement (z_3, Type B)
- Incidence angle of the measured distance under the bridge (z_4, Type B)
- Accuracy of the zenith angle (z_5, Type A) and the vertical index error (z_6, Type B)
- Vertical resolution for the zenith angle (the step width of the motor) (z_7, Type B)

The uncertainties and the pdf / membership function for the input quantities z_i are given in Tab. 1. The assumptions for the uncertainties of z_1, z_5 and z_6 are based on the technical data from the manufacturer and for the uncertainties of z_2, z_3 and z_4 on (Schulz and Ingensand, 2004) and for z_7 on (Reshetyuk, 2006). The input quantities z_3 and z_4 have a correlation of 0.5. In order to have an easier representation, each input quantity is modelled either as random or as systematic. Please note that in general the uncertainty budget of each input quantity may consist of a random **and** systematic component.

Input quantity z_i	Error component	pdf / membership function	Uncertainty	Type
z_1	random	normal	$\sigma = 3\,mm$	A
z_2	systematic	triangular	$a_+ - a = 3\,mm$ $\tilde{z}_{\alpha=0,r} = 3\,mm$	B
z_3	random	normal	$\sigma = 0.2\,mm\,(1831)$ $\sigma = 0.9\,mm\,(8987)$	B
z_4	random	normal	$\sigma = 2.6\,mm\,(1831)$ $\sigma = 7.2\,mm\,(8987)$	B
z_5	random	normal	$\sigma = 20\,mgon$	A
z_6	systematic	triangular	$a_+ - a = 20\,mgon$ $\tilde{z}_{\alpha=0,r} = 20\,mgon$	B
z_7	systematic	rectangular	$a_+ - a = 10\,mgon$ $\tilde{z}_{\alpha=0,r} = 10\,mgon$	B

Table 1 - Uncertainties for the input quantities $\boldsymbol{z}$

4.2. Specification and Discussion of the Numerical Results

This study focuses on the comparison of two different techniques to model and propagate the occurring uncertainties in Tab. 1. The pdfs and the order of magnitude of the uncertainties from Tab. 1 are in our opinion realistic. Their description must be carefully examined in future work, but this is not part of the paper. The results for the numerical example are obtained by the techniques described in section 2 and 3.

4.2.1. Uncertainties obtained by the Monte Carlo approach

In the MC approach the random and systematic components from Tab. 1 are treated as having a random nature. According to section 2 we obtain the uncertainty and the confidence interval of the output quantity $y \cong w_{scan}(x,t)$ for $M = 100000$ runs as:

Monte Carlo result	Point 1831 (7.28 m)	Point 8987 (21.90 m)
$\hat{\sigma}_y$	$4.4\,mm$	$5.9\,mm$
$y_{conf,MC} = [\underline{y}, \overline{y}]$ with $\gamma = 2{,}5\%$	$[\text{-}8.6\,mm, 8.6\,mm]$	$[-11.6\,mm, 11.7\,mm]$

4.2.2. Uncertainties obtained by the Fuzzy approach

In the Fuzzy approach the treatment of the random and systematic component in the propagation process of the uncertainties is different, see section 3. Whereas the random part is treated with the law of propagation of covariances or wth the MC approach, systematic errors are propagated within a sensitivity analysis (see section 3.2). According to section 3.2 we obtain the uncertainty and the systematic component of the output quantity $y \cong w_{scan}(x,t)$ for $\alpha = 0$ and $\alpha = 1$ with Eq. (12c), (15) and (16):

Fuzzy result (systematic component)	Point 1831 (7.28 m)	Point 8987 (21.90 m)
$\tilde{y}_{\alpha=1,r} = (\tilde{y}_{\alpha=1,\max} - \tilde{y}_{\alpha=1,\min})/2$	$0.2\ mm$	$4.8\ mm$
$\tilde{y}_{\alpha=0,r} = (\tilde{y}_{\alpha=0,\max} - \tilde{y}_{\alpha=0,\min})/2$	$10.3\ mm$	$16.1\ mm$

The α-level of zero refers to the pessimistic case and the α-level of one to the optimistic case. Within the propagation process of the systematic component, the radius $\tilde{z}_{\alpha,r}$ of all random components z_i from Tab. 1 is zero. In the presented propagation process a systematic error component cannot be reduced by repeated measurements. The small systematic error for the Point 1831 is due to the small influence of the systematic errors of the zenith angle.

For the propagation process of the random components with the methods described in section 2.1, the uncertainty of the input quantities with a systematic error component is set to zero, and we obtain the uncertainty and the confidence interval of the output quantity $y \cong w_{scan}(x,t)$ for $M = 100000$ runs as:

Fuzzy result (random component)	Point 1831 (7.28 m)	Point 8987 (21.90 m)
$\hat{\sigma}_y$	$3.9\ mm$	$5.4\ mm$
$y_{conf,MC} = [\underline{y}, \overline{y}]$ with $\gamma = 2,5\%$	$[-7.6\ mm, 7.6\ mm]$	$[-10.6\ mm, 10.7\ mm]$

Finally, we obtain the confidence interval for the Fuzzy approach with Eq. (17) for $\alpha = 0$ and $\alpha = 1$ as:

Fuzzy result (confidence interval)	Point 1831 (7.28 m)	Point 8987 (21.90 m)
$y_{conf,Fuzzy} = [\underline{y} - \tilde{y}_{\alpha=1,r}\ ;\ \overline{y} + \tilde{y}_{\alpha=1,r}]$ for $\alpha = 1$	$[-7.8\ mm, 7.8\ mm]$	$[-15.4\ mm, 15.5\ mm]$
$y_{conf,Fuzzy} = [\underline{y} - \tilde{y}_{\alpha=0,r}\ ;\ \overline{y} + \tilde{y}_{\alpha=0,r}]$ for $\alpha = 0$	$[-17.9\ mm, 17.9\ mm]$	$[-26.7\ mm, 26.8\ mm]$

5. DISCUSSION AND OUTLOOK

In the Monte Carlo approach, the uncertainty of the systematic component can be reduced by averaged/repeated measurements. Therefore, it is a more optimistic representation of the uncertainties than in the Fuzzy approach, where the systematic component of the uncertainties cannot be reduced by averaged/repeated measurements.

Further work has to deal with an extended discussion of the presented Fuzzy approach with input quantities having both types of uncertainties, a random and systematic component. Additionally, the bias of the output quantity resulting from the evaluation of non-linear functions has to be discussed in detail, especially in the Fuzzy approach.

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References

Alkhatib, H. (2007): *On Monte Carlo methods with applications to the current satellite gravity missions*, PhD thesis, Institute of theoretical Geodesy, University of Bonn.

Bandemer, H. (2006): *Mathematics of Uncertainty – Ideas, Methods, Application Problems,* Springer, Berlin, Heidelberg, New York.

Buckland, S. T. (1984): *MC Confidence Intervals*, in: Biometrica. Vol. 40 No. 3, pp. 811-817.

Ferson, S.; Ginzburg, L.; Kreinovich, V.; Nguyen, H.T. and Starks, S.A. (2002): *Uncertainty in risk analysis: towards a general second-order approach combining interval, probabilistic, and fuzzy techniques*, in: Proceedings of the 2002 IEEE International Conference on Fuzzy Systems, pp. 1342-1347.

Gentel, J.E. (2003): *Random Number Generation and Monte Carlo Methods*. Springer, Berlin.

ISO (1995): *Guide to the expression of uncertainty in measurements (GUM)*. International Organization for Standardization, Geneva, 1995.

Koch, K. R. (2007): *Introduction to Bayesian Statistics*. 2nd Edition, Springer, Berlin.

Kutterer, H. and Neumann, I. (2007): *Multidimensional statistical tests for imprecise data*. In: Xu, P., Liu, J. and Dermanis, A. (Eds.): Proceedings of the 6. Hotine-Marussi-Symposium, Association of Geodesy, Vol. 132, Springer, New York, pp. 232-237.

Möller, B. and Beer, M. (2004): *Fuzzy Randomness -Uncertainty in Civil Engineering and Computational Mechanics-*, Springer, Berlin and New York.

Reshetyuk, Y. (2006): *Investigation and calibration of pulsed time-of-flight terrestrial laser scanners*. Licentiate thesis in Geodesy, Royal Institute of Technology (KTH), Department of Transport and Economics, Stockholm, Sweden.

Schulz, T. and Ingensand, H. (2004): *Influencing Variables, Precision and Accuracy of Terrestrial Laser Scanners.* Intergeo East, Bratislava, Slovakia.

Strübing, T. (2007): *Realization of a sensor integrated measuring system for the acquisition of deformation processes of structures* (in German), diploma thesis (unpublished), Geodetic Institute, Leibniz University of Hannover

Viertl, R. (1996): *Statistical Methods for Non-Precise DATA*, CRC Press, New York London.

Zadeh, L.A. (1965): *Fuzzy sets,* in: Information Control, Vol. 8, pp. 338-353.

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