Accuracy of M_{split} estimates in the context of vertical displacement analysis

Patrycja Wyszkowska[1], Robert Duchnowski[2]

[1] Institute of Geodesy, University of Warmia and Mazury in Olsztyn, 1 Oczapowskiego St., 10-957 Olsztyn, Poland, (patrycja.wyszkowska@uwm.edu.pl)

[2] Institute of Geodesy, University of Warmia and Mazury in Olsztyn, 1 Oczapowskiego St., 10-957 Olsztyn, Poland, (robert.duchnowski@uwm.edu.pl)

Key words: *Vertical displacement*; *M_{split} estimation*; *accuracy; deformation analysis; Monte Carlo simulations*

ABSTRACT

M_{split} estimation is a development of M-estimation which can also be applied in deformation analysis. This paper concerns two types of such an estimation: squared M_{split} estimation (SMS), which is based on the assumption of normality of the observation errors and absolute M_{split} estimation (AMS), which is based on L_1 norm criterion. The main objective of the paper is to assess the accuracy of such estimators in the cases of analysis of vertical displacements of network points (or object points). To achieve the main objective of the paper we apply the Monte Carlo simulations. Another interesting issue is to compare the accuracy of SMS estimation with the accuracy of AMS estimation but also with the accuracy of the traditional least squares estimation (LS). Generally, AMS estimates have a better accuracy than SMS estimates. In many cases the accuracy of both M_{split} estimates is similar to the accuracy of LS estimates. However, if there are some nonrandom errors in the observation sets then there are some cases when the accuracy of AMS estimates is better than the accuracy of the rest of the estimates considered here. It stems from the fact that AMS estimates are robust against disturbances which have a small magnitude. It is also worth noting than the accuracy of both M_{split} estimates might also depend on the values of the point displacements; however, such an influence might be varied for different network points or network shapes themselves. The paper presents some empirical analysis in such a context.

I. INTRODUCTION

M_{split} estimation is a development of M-estimation. That method assumes that an observation set is unrecognized mixture of realizations of different random variables (Wiśniewski, 2009, 2010). In the case of that method, there is a split of the functional model and it means that each observation can be a realization of either of some different random variables. The process of assignment of the observations to the particular variable is automatic during iterative adjustment (also to the particular variant of the split functional model). On the whole, the most common variant of M_{split} estimators concerns two competitive parameters that stems from the split of the functional model into two competitive ones, $q = 2$. Obviously, there are also variants for bigger number of competitive functional models q, which are called as $M_{split(q)}$ estimates. Generally, the most popular method so far is the squared M_{split} estimation which is based on the objective function which stems from the assumption of normality of the observation errors. The basic properties of the squared M_{split} estimation has already been investigated. Up to now, the squared M_{split} estimates are applied in, e.g., deformation analyses (Duchnowski and Wiśniewski, 2011; Zienkiewicz, 2015; Zienkiewicz and Baryła, 2015), robust coordinate transformation (Janicka and Rapiński, 2013), elaboration of the data sets from the terrestrial laser scanning or the systems of Airborne Laser Scanning (Błaszczak-Bąk *et al.,* 2015; Janowski and Rapiński, 2013) and direct identification of gross errors (Li *et al.*, 2013). In turn, a modification of M_{split} estimation, namely Shift-M_{split} estimation is used in a direct estimation of the parameter differences (Duchnowski and Wiśniewski, 2012; Wiśniewski and Zienkiewicz, 2016) and it might be an alternative for some other natural robust estimation of such a shift, e.g., the Hodges-Lehmann estimation (Duchnowski, 2013; Duchnowski and Wiśniewski, 2014, 2017; Hodges and Lehmann, 1963; Huber and Ronchetti, 2009).

The theoretical foundations of M_{split} estimation allow us to develop the method and design different objective functions. The new variants of M_{split} estimation might have different properties than the squared M_{split} estimation because of the other type of the objective function. One of the possible ways to design such an alternative variant of M_{split} estimation is to base the objective function on the functions of the traditional robust methods. Thus, in this paper we consider such an alternative variant of M_{split} estimation which objective function is based on an application of L_1 norm condition (or the least absolute deviations, LAD), namely the absolute M_{split} estimation (see, Wyszkowska and Duchnowski, 2019). From the practical point of view, we should know the accuracy of both variants of M_{split} estimator. Thus, the main objective of the paper is to assess such an accuracy. Since we are especially interested in application of the methods in question in deformation analysis, thus we assess the estimation accuracy in the context of vertical displacement analysis. Note, that the theory of how to assess the accuracy of M_{split} estimators is not derived so far (especially with respect to the absolute variant), thus we will apply an empirical approach to achieve the paper goal. With regard to accuracy, we will use both root-mean-square deviation (RMSD) and standard deviation (SD). All empirical tests are based on Monte Carlo simulations.

II. THEORETICAL FOUNDATIONS

In the case of M_{split} estimation a classical linear model of observations:

$$\mathbf{y} = \boldsymbol{\theta} + \mathbf{v} = \mathbf{AX} + \mathbf{v} \tag{1}$$

where **y** = observation vector
θ = location parameter vector
v = measurement errors vector
A = known rectangular matrix
X = unknown parameter vector.

is split into q competitive models (Wiśniewski, 2009; 2010):

$$\mathbf{y} = \boldsymbol{\theta}_{(l)} + \mathbf{v}_{(l)} = \mathbf{AX}_{(l)} + \mathbf{v}_{(l)} \tag{2}$$

where $l = 1, ..., q$ and it is the number of the competitive model.

It means that the split functional models concern the same observation y_i. Due to the split of functional models we obtain q competitive versions of parameters. Thus, one can say that the observation set is an unknown mixture of realizations of q random variables which differ from each other at least in the location parameters. Generally, we do not know how particular observation is assigned to a respective parameter version. To put it another way, one has no information how the observations should be divided into q groups. The assumption about number of competitive models depends on the analyst's knowledge and experience and/or particular estimation problem (Zienkiewicz, 2018).

Here, we assume that $q = 2$, so the functional models can be written as:

$$\begin{aligned} \mathbf{y} &= \boldsymbol{\theta}_{(1)} + \mathbf{v}_{(1)} = \mathbf{AX}_{(1)} + \mathbf{v}_{(1)} \\ \mathbf{y} &= \boldsymbol{\theta}_{(2)} + \mathbf{v}_{(2)} = \mathbf{AX}_{(2)} + \mathbf{v}_{(2)} \end{aligned} \tag{3}$$

M_{split} estimation is characterized by the optimization problem, which is defined by following formula (more information about the general optimization problem can be found in Wiśniewski, 2009; 2010):

$$\varphi(\mathbf{X}_{(1)}, \mathbf{X}_{(2)}) = \sum_{i=1}^{n} \rho(v_{i(1)})\rho(v_{i(2)}) = \min_{\mathbf{X}_{(1)}, \mathbf{X}_{(2)}} \tag{4}$$

where ρ = chosen function which defines the objective function.

Generally, the Newton method can be applied to find the estimates $\hat{\mathbf{X}}_{(1)}, \hat{\mathbf{X}}_{(2)}$ which minimize that function. Additionally, it is worth noting that the properties of the method stem from the two following characteristic functions (e.g., Wiśniewski, 2009):

$$\begin{aligned} \psi(v_{(1)}) &= \frac{d\rho(v_{(1)})\rho(v_{(2)})}{dv_{(1)}} = \frac{\rho(v_{(2)})d\rho(v_{(1)})}{dv_{(1)}} \\ \psi(v_{(2)}) &= \frac{d\rho(v_{(1)})\rho(v_{(2)})}{dv_{(2)}} = \frac{\rho(v_{(1)})d\rho(v_{(2)})}{dv_{(2)}} \end{aligned} \tag{5}$$

$$w(v_{(1)}) = \frac{\psi(v_{(1)})}{v_{(1)}} \quad \text{and} \quad w(v_{(2)}) = \frac{\psi(v_{(2)})}{v_{(2)}} \tag{6}$$

where ψ = influence function
w = weight function.

The weight function is also very useful in designing numerical algorithms leading to the sought estimates. In the case of the first considered variant of M_{split} estimation, namely the squared M_{split} estimation (SMS estimation), the objective function is as follows (Wiśniewski, 2009):

$$\varphi(\mathbf{X}_{(1)}, \mathbf{X}_{(2)}) = \sum_{i=1}^{n} \rho(v_{i(1)})\rho(v_{i(2)}) = \sum_{i=1}^{n} v_{i(1)}^2 v_{i(2)}^2 \tag{7}$$

Thus, the respective influence functions are defined as:

$$\psi(v_{(1)}) = 2v_{(1)}v_{(2)}^2 \quad \text{and} \quad \psi(v_{(2)}) = 2v_{(1)}^2 v_{(2)} \tag{8}$$

and the respective weight functions as:

$$w(v_{(1)}) = \frac{\psi(v_{(1)})}{2v_{(1)}} = v_{(2)}^2 \quad \text{and} \quad w(v_{(2)}) = \frac{\psi(v_{(2)})}{2v_{(2)}} = v_{(1)}^2 \tag{9}$$

Here, another considered variant of M_{split} estimation is the absolute M_{split} estimation (AMS estimation) (see, Wyszkowska and Duchnowski, 2019), which is developed on the basis of the objective function of LAD method (e.g., Baselga and García-Asenjo, 2008; Marshall and Bethel, 1996):

$$\varphi_{LAD}(\mathbf{X}) = \sum_{i=1}^{n} \rho(v_i) = \sum_{i=1}^{n} |v_i| \tag{10}$$

Considering such a function, one can propose the objective function of AMS estimation in the following form:

$$\varphi(\mathbf{X}_{(1)},\mathbf{X}_{(2)})=\sum_{i=1}^{n}\rho(v_{i(1)})\rho(v_{i(2)})=\sum_{i=1}^{n}|v_{i(1)}||v_{i(2)}| \quad (11)$$

This leads to respectively influence functions:

$$\begin{aligned}\psi_{(1)}(v_{(1)},v_{(2)})&=\begin{cases}-|v_{(2)}| & for\ v_{(1)}<0\\ |v_{(2)}| & for\ v_{(1)}>0\end{cases}\\ \psi_{(2)}(v_{(1)},v_{(2)})&=\begin{cases}-|v_{(1)}| & for\ v_{(2)}<0\\ |v_{(1)}| & for\ v_{(2)}>0\end{cases}\end{aligned} \quad (12)$$

and weight functions:

$$\begin{aligned}w_{(1)}(v_{(1)},v_{(2)})&=\begin{cases}-\dfrac{|v_{(2)}|}{2v_{(1)}} & for\ v_{(1)}<0\\ \dfrac{|v_{(2)}|}{2v_{(1)}} & for\ v_{(1)}>0\end{cases}\\ w_{(2)}(v_{(1)},v_{(2)})&=\begin{cases}-\dfrac{|v_{(1)}|}{2v_{(2)}} & for\ v_{(2)}<0\\ \dfrac{|v_{(1)}|}{2v_{(2)}} & for\ v_{(2)}>0\end{cases}\end{aligned} \quad (13)$$

Generally, M_{split} estimation is an iterative process (Wiśniewski, 2009). In the case of AMS estimation the weight functions depend on the both $v_{(1)}$ and $v_{(2)}$, so the subsequent parameter estimates $\hat{\mathbf{X}}_{(1)},\hat{\mathbf{X}}_{(2)}$ must be estimated as $\left(\hat{\mathbf{X}}_{(1)}^{j-1},\hat{\mathbf{X}}_{(2)}^{j-1}\right)\Rightarrow\left(\hat{\mathbf{X}}_{(1)}^{j},\hat{\mathbf{X}}_{(2)}^{j}\right)$. SMS estimates can also be computed in the same way; however, there is also another way given by Wiśniewski (2009) where $\hat{\mathbf{X}}_{(1)}^{j-1}\Rightarrow\hat{\mathbf{X}}_{(2)}^{j-1}\Rightarrow\hat{\mathbf{X}}_{(1)}^{j}\Rightarrow\hat{\mathbf{X}}_{(2)}^{j}$. Here, we will apply the first way only. Usually, the least square estimate (LS), namely $\hat{\mathbf{X}}_{LS}$, is also the starting point of the iterative processes, hence $\hat{\mathbf{X}}^{0}=\hat{\mathbf{X}}_{LS}$ (Wiśniewski, 2009; 2010). However, in the case of AMS estimation, the starting values of $\hat{\mathbf{X}}_{(1)},\hat{\mathbf{X}}_{(2)}$ cannot be the same because of the failure of the start of iterative process. The basic solution in such a case is to assume:

$$\begin{aligned}\hat{\mathbf{X}}_{(1)}^{0}&=\hat{\mathbf{X}}_{LS}+\Delta\\ \hat{\mathbf{X}}_{(2)}^{0}&=\hat{\mathbf{X}}_{LS}-\Delta\end{aligned} \quad (14)$$

where: Δ = assumed positive value.

The iterative process itself can be written as follows (e.g., Wiśniewski, 2009; Wyszkowska and Duchnowski, 2019)

$$\begin{aligned}\mathbf{X}_{(1)}^{j}&=\mathbf{X}_{(1)}^{j-1}-d\mathbf{X}_{(1)}^{j}\\ d\mathbf{X}_{(1)}^{j}&=\left[\mathbf{H}_{(1)}\left(\mathbf{X}_{(1)}^{j-1},\mathbf{X}_{(2)}^{j-1}\right)\right]^{-1}\mathbf{g}_{(1)}\left(\mathbf{X}_{(1)}^{j-1},\mathbf{X}_{(2)}^{j-1}\right)\\ \mathbf{X}_{(2)}^{j}&=\mathbf{X}_{(2)}^{j-1}-d\mathbf{X}_{(2)}^{j}\\ d\mathbf{X}_{(2)}^{j}&=\left[\mathbf{H}_{(2)}\left(\mathbf{X}_{(1)}^{j-1},\mathbf{X}_{(2)}^{j-1}\right)\right]^{-1}\mathbf{g}_{(2)}\left(\mathbf{X}_{(1)}^{j-1},\mathbf{X}_{(2)}^{j-1}\right)\end{aligned} \quad (15)$$

where $d\mathbf{X}$ = increment to parameter
$\mathbf{H}$ = Hessian matrix
$\mathbf{g}$ = gradient

and

$$\begin{aligned}&\begin{cases}\mathbf{H}_{(1)}\left(\mathbf{X}_{(1)}^{j-1},\mathbf{X}_{(2)}^{j-1}\right)=2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1},\mathbf{v}_{(2)}^{j-1}\right)\mathbf{A}\\ \mathbf{g}_{(1)}\left(\mathbf{X}_{(1)}^{j-1},\mathbf{X}_{(2)}^{j-1}\right)=-2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1},\mathbf{v}_{(2)}^{j-1}\right)\mathbf{v}_{(1)}^{j-1}\end{cases}\\ &\begin{cases}\mathbf{H}_{(2)}\left(\mathbf{X}_{(1)}^{j-1},\mathbf{X}_{(2)}^{j-1}\right)=2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j-1},\mathbf{v}_{(2)}^{j-1}\right)\mathbf{A}\\ \mathbf{g}_{(2)}\left(\mathbf{X}_{(1)}^{j-1},\mathbf{X}_{(2)}^{j-1}\right)=-2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j-1},\mathbf{v}_{(2)}^{j-1}\right)\mathbf{v}_{(2)}^{j-1}\end{cases}\end{aligned} \quad (16)$$

where $\mathbf{w}$ = respective weight matrix.

The weight matrices $\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1},\mathbf{v}_{(2)}^{j-1}\right)$ and $\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j-1},\mathbf{v}_{(2)}^{j-1}\right)$ in the iterative process can be computed as:

$$\begin{aligned}\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1},\mathbf{v}_{(2)}^{j-1}\right)&=\mathrm{Diag}\left[w_{(1)}(v_{1(1)}^{j-1},v_{1(2)}^{j-1}),\ldots,w_{(1)}(v_{n(1)}^{j-1},v_{n(2)}^{j-1})\right]\\ \mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j-1},\mathbf{v}_{(2)}^{j-1}\right)&=\mathrm{Diag}\left[w_{(2)}(v_{1(1)}^{j-1},v_{1(2)}^{j-1}),\ldots,w_{(2)}(v_{n(1)}^{j-1},v_{n(2)}^{j-1})\right]\end{aligned} \quad (17)$$

That approach is successfully applied in SMS estimation. Notwithstanding this, in the case of AMS estimation, there is a possible problem with such an iterative process. It concerns singularity resulting from the weight functions of Eq. (13) which are not defined for measurement errors $v_{(l)}$ equal 0. In the situation when $v_{(l)}$ aims for zero, then the weight function $w_{(l)}(v_{(1)},v_{(2)})$ goes up to infinity. Thus, we should use a modified weight functions of Eq. (13), namely:

$$\begin{aligned}w_{(1)}^{*}(v_{(1)},v_{(2)})&=\begin{cases}\dfrac{|v_{(2)}|}{2|v_{(1)}|} & for\ |v_{(1)}|\geq d\\ \dfrac{|v_{(2)}|}{2d} & for\ |v_{(1)}|<d\end{cases}\\ w_{(2)}^{*}(v_{(1)},v_{(2)})&=\begin{cases}\dfrac{|v_{(1)}|}{2|v_{(2)}|} & for\ |v_{(2)}|\geq d\\ \dfrac{|v_{(1)}|}{2d} & for\ |v_{(2)}|<d\end{cases}\end{aligned} \quad (18)$$

where w^{*} = modification of weight functions
d = assumed positive constant.

On the whole, the iterative process should be finished for such $j = k$, when $\mathbf{g}_{(1)}\left(\mathbf{X}_{(1)}^{k-1},\mathbf{X}_{(2)}^{k-1}\right)=0$, $\mathbf{g}_{(2)}\left(\mathbf{X}_{(1)}^{k-1},\mathbf{X}_{(2)}^{k-1}\right)=0$ and thereupon $\hat{\mathbf{X}}_{(1)}=\mathbf{X}_{(1)}^{k}=\mathbf{X}_{(1)}^{k-1}$, $\hat{\mathbf{X}}_{(2)}=\mathbf{X}_{(2)}^{k}=\mathbf{X}_{(2)}^{k-1}$.

III. Empirical tests

Let us consider an example of deformation analysis and simulate a levelling network which consists of two reference points P_1 and P_2 and three object points A, B and C (Figure 1). We assume that all observations (eight height differences) are measured at two measurement epochs, thus $q = 2$. Note, that we have one combined observation vector here (that vector contains observations from both epochs).

Due to the above, one can have two competitive models of Eq. (3), where:

$$\mathbf{A}=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}^T \quad (19)$$

$$\mathbf{X}_{(1)}=\begin{bmatrix} H_A^I \\ H_B^I \\ H_C^I \end{bmatrix} \text{ and } \mathbf{X}_{(2)}=\begin{bmatrix} H_A^{II} \\ H_B^{II} \\ H_C^{II} \end{bmatrix} \quad (20)$$

where H^I = theoretical height at the first measurement epoch
H^{II} = theoretical height at the second measurement epoch.

Hence, there is an opportunity to determine vertical point displacement from the following formula:

$$\mathbf{\Delta X}=\mathbf{X}_{(2)}-\mathbf{X}_{(1)}=\begin{bmatrix} H_A^{II} \\ H_B^{II} \\ H_C^{II} \end{bmatrix}-\begin{bmatrix} H_A^{I} \\ H_B^{I} \\ H_C^{I} \end{bmatrix}=\begin{bmatrix} \Delta H_A \\ \Delta H_B \\ \Delta H_C \end{bmatrix} \quad (21)$$

where $\mathbf{\Delta X}$ = parameter difference
ΔH = vertical point displacement.

All empirical analyses are based on Crude Monte Carlo method (here denoted as MC) which is a statistical sampling approach. The Monte Carlo method is very useful tool to solve many problems which might occur in geodesy and surveying (e.g., Duchnowski and Wiśniewski, 2014; Wyszkowska, 2017; Wyszkowska and Duchnowski, 2018, 2019). We assume that observations are independent, and their errors are normally distributed $v_i \sim N(0,\sigma^2)$, where standard deviation of measurement errors $\sigma = 1\,\text{mm}$. Another

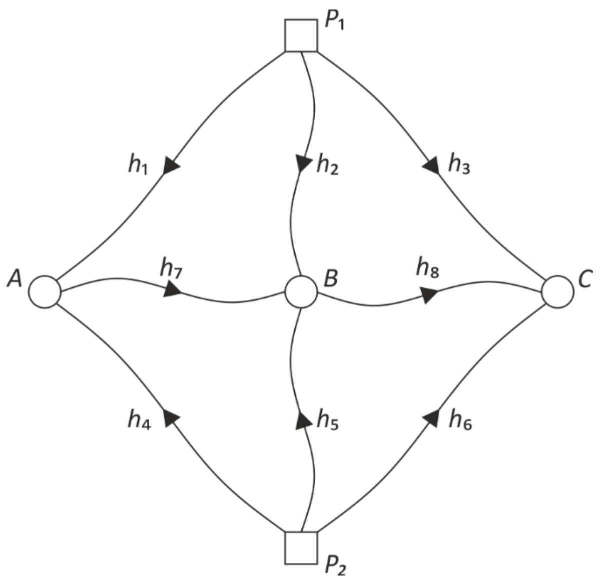

Figure 1. Simulated levelling network

assumption is that we execute 1000 simulations in all empirical tests. That number of simulations seems enough for purposes of the present paper. All computations were carried out in Mathcad 15.0. Thanks to the MC method it is possible to assess the accuracy of estimates by computing empirical standard deviations and root-mean-square deviations of these distributions (e.g., Duchnowski and Wiśniewski, 2014, 2017; Duchnowski and Wyszkowska, 2017; Wyszkowska, 2017):

$$\text{SD}=\sqrt{\sum_{i=1}^{n}\frac{\left(\hat{X}_i^{MC}-\bar{X}^{MC}\right)^2}{n}} \quad (22)$$

$$\text{RMSD}=\sqrt{\sum_{i=1}^{n}\frac{\left(\hat{X}_i^{MC}-X\right)^2}{n}} \quad (23)$$

where SD = empirical standard deviation
$\hat{X}_i^{MC}$ = estimated value at the ith simulation
$\bar{X}^{MC}$ = mean value of the parameter from all Monte Carlo simulations
n = number of simulations
RMSD = root-mean-square deviation
X = theoretical value of the estimated parameter.

Without loss of generality we can assume that $H_{P_1}^I = H_{P_1}^{II} = 0\,\text{mm}$ and $H_{P_2}^I = H_{P_2}^{II} = 0\,\text{mm}$ at both measurement epochs. In other words, the reference points are stable. We can also make the next assumption that the theoretical heights of the object points $H_A^I = 0\,\text{mm}$, $H_B^I = 0\,\text{mm}$ and $H_C^I = 0\,\text{mm}$ at the first measurement epoch. Now, let us consider the following several variants to investigate the accuracy of all estimates considered here.

First, let $H_B^{II} = 0\,\text{mm}$, $H_C^{II} = 0\,\text{mm}$ and let H_A^{II} might vary within the interval $< 0\,\text{mm}, 50\,\text{mm} >$. Thus, we are checking if the accuracy of SMS or AMS estimates depend on the magnitude of the vertical point displacements. Figure 2 presents RMSDs obtained for all estimates. Note, that AMS estimation gives superior results than SMS estimation; however, the smallest RMSDs, which are close to 1 mm for all point displacements, are obtained for LS estimation. There are some disturbances for RMSD(ΔH_A), when the values of ΔH_A are quite small. It is probably caused by some problems with assigning observations to the appropriate measurement epochs when the parameter values are close to each other at both measurement epochs. We also compute SDs for that case of points displacements. Since they are close to the values of RMSDs, they are not shown here.

Now, let us assume that $H_B^{II} = 20\,\text{mm}$, $H_C^{II} = 50\,\text{mm}$. Figure 3 shows RMSDs for the second case of points displacements. The values of RMSD(ΔH_B) and RMSD(ΔH_C) are close to 1 mm for each considered method and for all ΔH_A. Once again, RMSD(ΔH_A) has much bigger values for both SMS and AMS estimates for relatively small ΔH_A (with respect to LS estimate); however, RMSD(ΔH_A) for all three methods are close to each other for bigger ΔH_A. Thus, the accuracy of M_{split} estimates depends on the point displacements. Once again, the computed SDs are similar to RMSDs, respectively, hence they are not presented here.

The third and fourth examples considered here concern the situation when the observation set from the second variant is disturbed by one gross error at the second measurement epoch. We assume that the height difference h_1^{II} is an outlier, and that it is disturbed by either positive or negative gross errors equal to 5 mm and -5 mm, respectively. Let us analyze the case with positive gross error. Respective RMSDs are presented in Figure 4 and SDs in Figure 5. When there is the outlier in observation set, the values of RMSD(ΔH_A) obtained for LS estimation are distant from these ones from the previous case. Furthermore, RMSDs(ΔH_A) for AMS estimation are usually better than those for LS and SMS methods. The exceptions are results obtained for such ΔH_A which are close to ΔH_B. Then RMSDs(ΔH_A) for AMS estimates are significantly bigger. That results from the coincidence between ΔH_A and ΔH_B. Note, that if ΔH_B ≈ ΔH_A then the observed (or simulated) $h_7^{II} \approx h_7^{I}$. It might happen that the

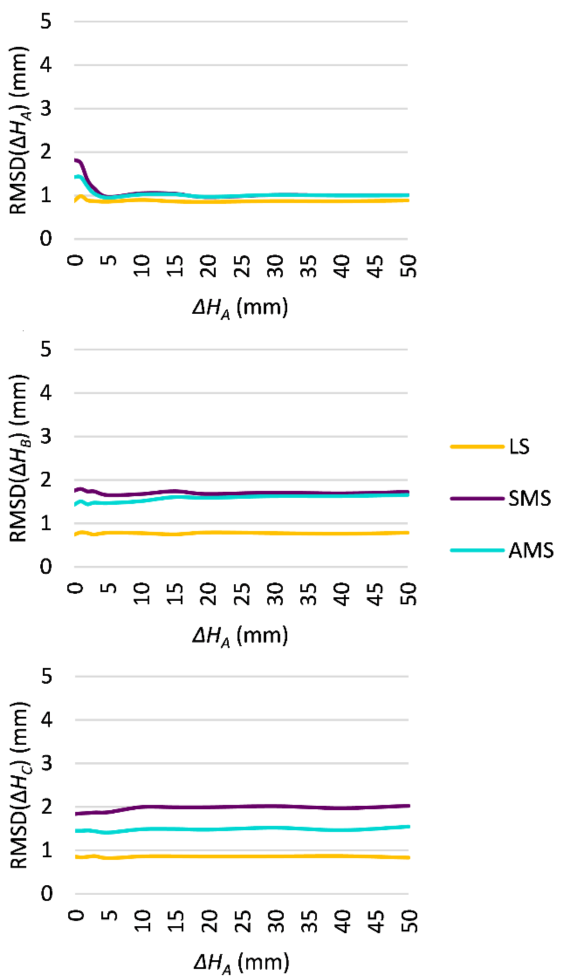

Figure 2. RMSDs of point displacements for different values of ΔH_A; where ΔH_B = 0 mm, ΔH_C = 0 mm

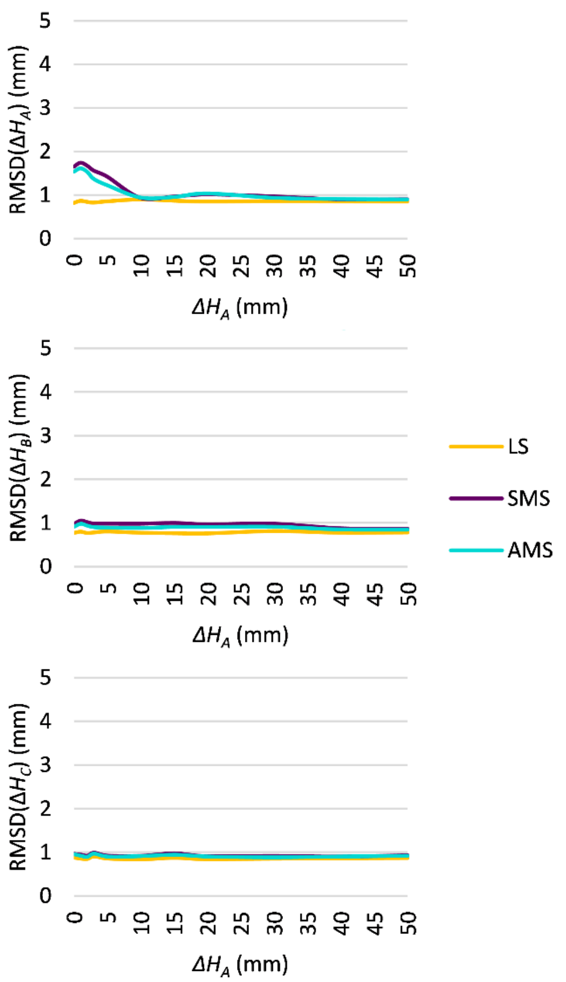

Figure 3. RMSDs of point displacements for different values of ΔH_A; where ΔH_B = 20 mm, ΔH_C = 50 mm

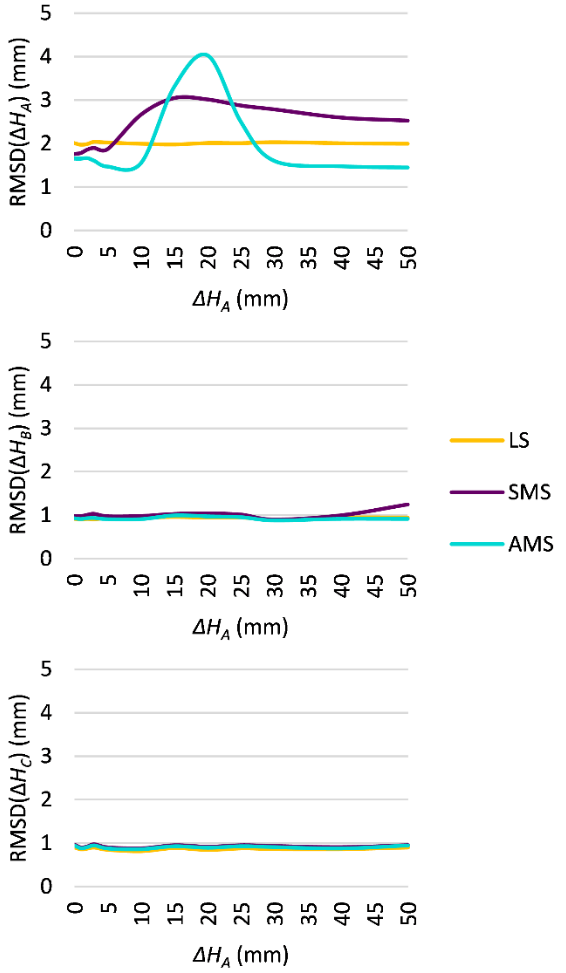

Figure 4. RMSDs of point displacements for different values of ΔH_A; where ΔH_B = 20 mm, ΔH_C = 50 mm with an observation affected by a positive gross error

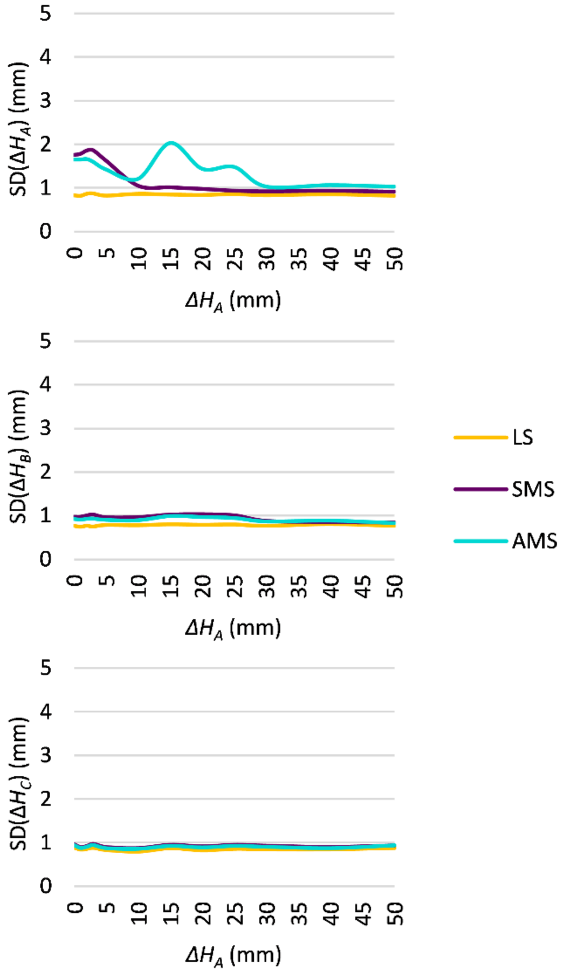

Figure 5. SDs of point displacements for different values of ΔH_A; where ΔH_B = 20 mm, ΔH_C = 50 mm with an observation affected by a positive gross error

simulated random error of either of h_7 coincides well with the outlying observation h_1^{II}. Such a h_7 would be assigned to the second measurement epoch which results in too high value of the estimated H_A^{II} and hence too high value of the estimated ΔH_A. In such a context, the coincidence between ΔH_A and ΔH_B (or any neighboring network points) might be defined as a problem of AMS estimation. Despite this, AMS estimation seems to be more robust against outlier than SMS estimation (RMSDs are generally smaller for AMS estimates than for SMS estimates). To compare accuracy of both estimates we also compute SDs. Obviously, SD(ΔH_A) of LS estimate is close to that from the previous case, respectively. Also, this time, if ΔH_A are relatively small then we observe the lower accuracy of both M_{split} estimates. Moreover, if there is a coincidence between ΔH_A and ΔH_B then SD(ΔH_A) for AMS estimation is also bigger. Whereas, the values of RMSD(ΔH_B), RMSD(ΔH_C) as well as SD(ΔH_B), SD(ΔH_C), respectively, are rather close to each other for each considered method.

Let us consider the last case with an observation h_1^{II} affected by a negative gross error. RMSDs of that case can be seen in Figure 6 and SDs in Figure 7, respectively. Note, that all RMSDs of LS estimates are close to these in Figure 4. On the whole, the superior results of RMSDs are obtained for AMS estimation. In other words, AMS estimation is less sensitive to such an outlier than SMS estimation and LS estimation are. The accuracy of SMS estimation is quite comparable to accuracy of AMS estimation for $\Delta H_A \leq 10$ mm (for both RMSDs or SDs). In turn, the bigger is ΔH_A, the larger is the difference between RMSDs(ΔH_A) of SMS and AMS estimates. Generally, the values of RMSDs(ΔH_A) and SDs(ΔH_A) of AMS estimation are close to each other. Note, that the relations between RMSD(ΔH_B), SD(ΔH_B) and RMSD(ΔH_C), SD(ΔH_C), respectively, are similar to the preceding case. Another interesting issue is that all values of SDs of that case are comparable to RMSDs which were obtained in the second case (no gross error case), so the occurrence of the gross error does not make SDs worse. It is noteworthy that this time the coincidence between ΔH_A and ΔH_B has no bad influence on the accuracy of both M_{split} estimates. That is due to the general properties of M_{split} estimation. Note, that the negative gross error of a moderate magnitude will place the outlier h_1^{II} "between" two measurement

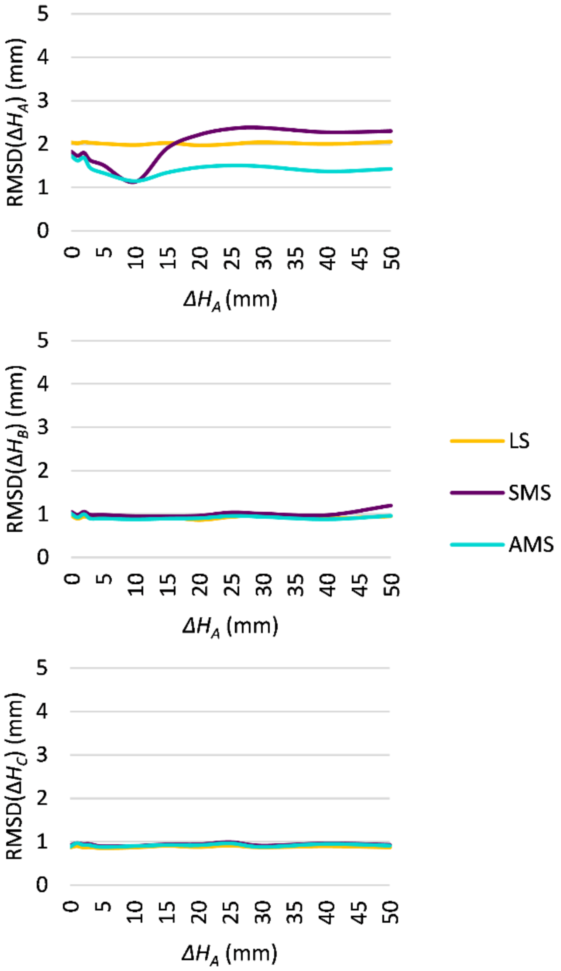

Figure 6. RMSDs of point displacements for different values of ΔH_A; where ΔH_B = 20 mm, ΔH_C = 50 mm with an observation affected by a negative gross error

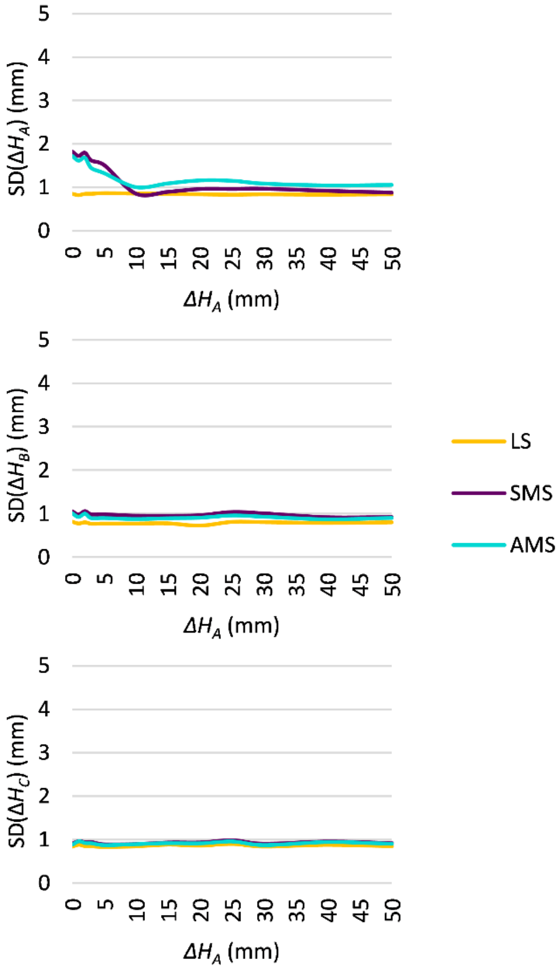

Figure 7. SDs of point displacements for different values of ΔH_A; where ΔH_B = 20 mm, ΔH_C = 50 mm with an observation affected by a negative gross error

epochs (in contrast to the previous case where the positive gross error locates the outlier "outside" the epochs). Such a location is usually not a problem especially for AMS method which seems robust against such outliers (Wyszkowska and Duchnowski, 2019).

IV. Conclusions

The paper presents an investigation of the accuracy of two variants of M_{split} estimates, namely the squared M_{split} estimates and the absolute M_{split} estimates in the context of the vertical deformation analysis in comparison to the accuracy of the least square estimation. On the whole, AMS estimates have a better accuracy than SMS estimates. The tests, which consider the observation set without outliers, show that the accuracy of both M_{split} estimates might be similar to the accuracy of LS estimates. What is more, such tests prove that the accuracy of both M_{split} estimates might depend on the values of the point displacements; however, such an influence might be varied for different network points, an observation affected by the negative gross error or network shapes themselves. There is also noticeable, that the general problem of assessing the accuracy of M_{split} estimates is related to assignment the observations to particular measurement epochs (especially, for such values which are close to each other at both measurement epochs). Then the accuracy of the estimates in question is lower. It increases with the growing difference between the parameters. Note, that empirical assessments of estimator accuracy (especially RMSD) can provide also other useful information. In the context of the paper, high values of RMSD mean that the point displacements are not estimated correctly by the method under study.

Whereas the analysis of two empirical tests which concern the occurrence of an observation affected by gross errors shows that usually AMS estimates are superior to the rest of the estimates considered here. It stems from the fact that AMS estimates seems more robust against disturbances which have a small magnitude. However, if the values of vertical displacements of two object points are close to each other, the accuracy would be distinctly lower. After all, it seems that the higher robustness against outliers (in most of the cases) makes the method an interesting alternative to well-known robust estimations and an advisable solution in some problems of surveying and geodesy.

V. ACKNOWLEDGEMENTS

This work was supported by Institute of Geodesy, University of Warmia and Mazury in Olsztyn [statutory research no. 28.610.002-300].

References

Baselga, S., and L. García-Asenjo (2008). GNSS differential positioning by robust estimation. *Journal of Surveying Engineering,* Vol. 134, No. 1, pp. 21-25.

Błaszczak-Bąk, W., A. Janowski, W. Kamiński, and J. Rapiński (2015). Application of the M_{split} method for filtering airborne laser scanning data-sets to estimate digital terrain models. *International Journal of Remote Sensing*, Vol. 36, No. 9, pp. 2421-2437.

Duchnowski, R. (2013). Hodges-Lehmann estimates in deformation analyses. *Journal of Geodesy*, Vol. 87, No. 10, pp. 873-884.

Duchnowski, R., and Z. Wiśniewski (2011). Shift-M_{split} estimation. *Geodesy and Cartography*, Vol. 60, No. 2, pp. 79-97.

Duchnowski, R., and Z. Wiśniewski (2012). Estimation of the shift between parameters of functional models of geodetic observations by applying M_{split} estimation. *Journal of Surveying Engineering*, Vol. 138, No. 1, pp. 1-8.

Duchnowski, R., and Z. Wiśniewski (2014). Comparison of two unconventional methods of estimation applied to determine network point displacement. *Survey Review*, Vol. 46, No. 339, pp. 401-405.

Duchnowski, R., and Z. Wiśniewski (2017). Accuracy of the Hodges–Lehmann estimates computed by applying Monte Carlo simulations. *Acta Geodaetica et Geophysica*, Vol. 52, No. 4, pp. 511-525.

Duchnowski, R., and P. Wyszkowska (2017). Leptokurtosis of error distribution and its influence on estimation accuracy. The case of three estimates applied in adjustment of geodetic measurements. In: *Proc. of 2017 Baltic Geodetic Congress (BGC Geomatics)*, IEEE, Piscataway, NJ, pp. 237-241.

Hodges Jr., J.L., and E.L. Lehmann (1963). Estimates of location based on rank tests. *Annals of Mathematical Statistics*, Vol. 34, No. 2, pp. 598-611.

Huber, P.J., and E.M. Ronchetti (2009). Robust statistics. 2nd ed. Wiley, Hoboken.

Janicka, J., and J. Rapiński (2013). M_{split} transformation of coordinates. *Survey Review*, Vol. 45, No. 331, pp. 269-274.

Janowski, A., and J. Rapiński (2013). M-split estimation in laser scanning data modeling. *Journal of the Indian Society of Remote Sensing*, Vol. 41, No. 1, pp. 15-19.

Li, J., A. Wang, and W. Xinyuan. (2013). M_{split} estimate the relationship between LS and its application in gross error detection. *Mine Surveying (China)*, No. 2, pp. 57-59.

Marshall, J. and J. Bethel (1996). Basic concepts of L_1 norm minimization for surveying applications. *Journal of Surveying Engineering*, Vol. 122, No. 4, pp. 168-179.

Wiśniewski, Z. (2009). Estimation of parameters in a split functional model of geodetic observations (M_{split} estimation). *Journal of Geodesy*, Vol. 83, No. 2, pp. 105-120.

Wiśniewski, Z. (2010). $M_{split(q)}$ estimation: estimation of parameters in a multi split functional model of geodetic observations. *Journal of Geodesy*, Vol. 84, No. 6, pp. 355-372.

Wiśniewski, Z., and M.H. Zienkiewicz (2016). Shift-M^*_{split} estimation in deformation analyses. *Journal of Surveying Engineering*, Vol. 142, No. 4.

Wyszkowska, P. (2017). Propagation of uncertainty by Monte Carlo simulations in case of basic geodetic computations. *Geodesy and Cartography*, Vol. 66, No. 2, pp. 333-346.

Wyszkowska, P., and R. Duchnowski (2018). Increasing reliability of basic R-estimates in deformation analysis. In: *Proc. of 2018 Baltic Geodetic Congress (BGC Geomatics)*, IEEE, Piscataway, NJ, pp. 241-245.

Wyszkowska, P., and R. Duchnowski (2019). M_{split} estimation based on L_1 norm condition. *Journal of Surveying Engineering*, in press.

Zienkiewicz, M.H. (2015). Application of M_{split} estimation to determine control points displacements in networks with unstable reference system. *Survey Review*, Vol. 47, No. 342, pp. 174-180.

Zienkiewicz, M.H. (2018). Determination of an adequate number of competitive functional models in the square $M_{split(q)}$ estimation with the use of a modified Baarda's approach. *Survey Review*.

Zienkiewicz, M.H., and R. Baryła (2015). Determination of vertical indicators of ground deformation in the Old and Main City of Gdansk area by applying unconventional method of robust estimation. *Acta Geodynamica et Geomaterialia*, Vol. 12, No. 3, pp. 249-257.